The problem is to determine the possible motions of three point masses \(m_1\), \(m_2\), and \(m_3\) which attract each other according to Newton's law of inverse squares. It started with the perturbative studies of Newton himself on the inequalities of the lunar motion[1]. In the 1740s it was constituted as the search for solutions (or at least approximate solutions) of a system of ordinary differential equations by the works of Euler, Clairaut and d'Alembert (with in particular the explanation by Clairaut of the motion of the lunar apogee). Much developed by Lagrange, Laplace and their followers, the mathematical theory entered a new era at the end of the 19th century with the works of Poincaré and since the 1950s with the development of computers. While the two-body problem is integrable and its solutions completely understood (see [2],[AKN],[Al],[BP]), solutions of the three-body problem may be of an arbitrary complexity and are very far from being completely understood.

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### Equations
The following form of the equations of motion, using a force function \(U\) (opposite of potential energy), goes back to Lagrange, who initiated the general study of the problem: if \(\mathbf{r}_i\) is the position of body \(i\) in the Euclidean space \(\mathbb{R}^p\) (scalar product \(\langle,\rangle\) norm \(\|\cdot\|\)), \(\sum m_i \frac{d^2 \mathbf{r}_i}{dt^2} = \sum m_i \frac{\mathbf{r}_j - \mathbf{r}_i}{\|\mathbf{r}_j - \mathbf{r}_i\|^3} = \nabla U(\mathbf{x}),\) where the gradient is taken with respect to \(\mathbf{x} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\).
The equations are invariant under time translations, Galilean boosts and space isometries. This implies the conservation of

- **the total energy** \( \langle H, \cdot \rangle \),
- **the linear momentum** \( \langle P = \sum_{i=1}^{3} m_i \frac{d}{dt} \vec{r}_i(t) \mid dt \rangle \) (by an appropriate choice of a Galilean frame one can suppose that \( \langle P = 0 \rangle \) and that the center of mass is at the origin),
- **the angular momentum bivector** \( \langle C = \sum_{i=1}^{3} m_i \vec{r}_i \wedge \frac{d}{dt} \vec{r}_i(t) \mid dt \rangle \) (identified with a real number if \( p=2 \) and with a vector if \( p=3 \)).

If the motion takes place on a fixed line, \( \langle C = 0, \cdot \rangle \) on the other hand, if \( \langle C = \cdot, \cdot \rangle \) the motion takes place in a fixed plane (Dziobek).

The **reduction of symmetries** was first accomplished by Lagrange in his great 1772 *Essai sur le problème des trois corps*, where the evolution of mutual distances in the spatial problem is seen to be ruled by a system of order 7.

Finally, the homogeneity of the potential implies a **scaling invariance**: if \( x(t) \) is a solution, so is \( x_{\lambda}(t) = \lambda^{-\frac{2}{3}} x(\lambda t) \) for any \( \lambda > 0 \). Moreover \( \langle H(x_{\lambda}(t)) = \lambda^\frac{2}{3} H(x(t)) \rangle \) and
\[ \langle C(x_{\lambda}(t)) = \lambda^{-\frac{1}{3}} C(x(t)) \rangle \] it follows that \( \langle |H| \rangle C(x) \rangle \) is invariant under scaling:
\[ \langle |H| \rangle C(x_{\lambda}(t)) = \langle |H| \rangle C(x(t)) \]

**Homographic solutions**

A configuration \( x = (\vec{r}_1, \vec{r}_2, \vec{r}_3) \) is called a **central configuration** if it collapses homothetically on its center of mass \( \langle \vec{r}_{\text{G}} \mid \cdot \rangle \) defined by
\[ \langle \vec{r}_{\text{G}} = \frac{1}{M} \sum_{i=1}^{3} m_i \vec{r}_i \rangle \] when released without initial velocities (such a motion is called homothetic).

This means that there exists \( \langle \lambda < 0 \rangle \) such that \( \langle \sum_{j=1}^{3} m_j \frac{\vec{r}_j - \vec{r}_i}{||\vec{r}_j - \vec{r}_i||^3} = \lambda (\vec{r}_i - \vec{r}_{\text{G}}) \rangle \), which is equivalent to \( \sum_{j=1}^{3} m_j \left( \frac{1}{r_{ij}^3} + \frac{\lambda}{M} \right) (\vec{r}_j - \vec{r}_i) = 0 \) where \( r_{ij} = ||\vec{r}_j - \vec{r}_i|| \). For non collinear configurations, the two vectors \( \vec{r}_j - \vec{r}_i, j \neq i \) are linearly independent and so the coefficients in the last sum must vanish. It follows that \( x^0 \) must be equilateral, a result first proved by Lagrange in his 1772 memoir. Collinear **central configurations** of three bodies were already known to Euler in 1763: the ratio of the distances of the midpoint to the extremes is the unique real solution of an equation of the fifth degree whose coefficients depend on the masses.

Central configurations are also the ones with **homographic motions**, along which the configuration changes only by similarities and each body has a Keplerian motion (elliptic, parabolic or hyperbolic depending on the sign of the energy), with the same eccentricity \( \langle e \cdot \cdot \rangle \). In the elliptic case, represented in the animations, \( \langle e \in [0,1] \rangle \)

- when \( \langle e=1 \rangle \) the motion is homothetic;
- when \( \langle e=0 \rangle \) the motion is a **relative equilibrium**: the configuration changes only by isometries. For example, the Moon should be placed between the Earth and the Sun at approximately four times its actual distance from the Earth in order to be in (unstable) collinear equilibrium, while the **Greeks and Trojans** are most numerous near the two (stable) positions where they would form with the Sun and Jupiter an equilateral triangle.

![Figure 1: a homothetic solution (e = 1) (animation by R. Moeckel).](image1.png)

![Figure 2: a homographic solution (e = 0.9) (animation by R. Moeckel).](image2.png)

![Figure 3: a relative equilibrium (e = 0) (animation by R. Moeckel).](image3.png)

R. Moeckel's handwritten **Trieste notes** are a very good reference on central configurations and the stability of three body relative equilibria. In 2005, R. Moeckel proved that the **Saari conjecture** is true for 3 bodies in \( \langle \mathbb{R}^d, d^t(x,y) \rangle \) the relative equilibria are the only motions whose moment of inertia with respect to the center of mass (that is \( \langle I = |x(t) - x_{\text{G}}| \rangle \)) is constant.

Central configurations play a key part in the analysis of

- the topology of the invariant manifolds obtained by fixing the values of the first integrals (Birkhoff, Smale, Albouy, McCord-Meyer-Wang);
- the analysis of total collisions \( \langle I = |x - x_{\text{G}}|^2 \rangle \rangle t \to 0 \rangle \) and completely parabolic motions \( \langle K = y \frac{r}{dx_{G}}(d)dt | d^t t \to 0 \rangle \) as \( \langle y \infty \rangle \), where the renormalized configuration defined by the bodies entering the collision must tend to the set of central configurations (Sundman).
The astronomer's three-body problem: i) the planetary problem

This is the case where one mass is much larger than all the other ones and the solutions one considers are close to circular and coplanar Keplerian motions. The typical problem is the motion around the Sun (mass \(m_0\)) of the two big planets, Jupiter and Saturn (masses \(\mu m_1, \mu m_2\)) of the order of \((m_0/1000)\) which contain most of the mass of the planets in the solar system.

An *equation free* description of the principal features of the planetary and the lunar problems was given in [Ai][3] by Sir G.B. Airy[4]. Browsing through this book could help some people having a more friendly view on equations.

**Reduction to the general problem of dynamics**

When written in Poincaré's *heliocentric coordinates* \([P2]\), \(|X_0=0,Y_0=0|+|\mu y_1+\mu y_2,quad X_j=x_j-x_0,quad Y_j=y_j,quad j=1,2,|\) where \(|x_j=vec r_j, j=0,1,2,|\) are the positions and where \(|y_0=0,m_0,0, d vec r_0/dt, \mu y_j=m_j, d vec r_j/dt, j=1,2,|\) are the linear momenta, the equations take the form of a perturbation of a pair of uncoupled Kepler problems in \((R^3)^2,|\) More precisely, one reduces the translation symmetry by restricting to the value \(|Y_0=0|\) the total linear momentum and quotienting by translations. After dividing the new Hamiltonian and symplectic form by \(|\mu|\) one obtains the following Hamiltonian, defined on \(|(T^*R^3)^4equiv R^3(12)^2\) \(|(=(R^3)^4,\|\) coordinates \(|(X_1,X_2,Y_1,Y_2))|\) deprived of the collision set \(|(X_1=0) or (X_1=X_2))\) with its canonical symplectic structure: \(|H=\sum_{j=0}^{2} left( (L_1)^2 (Y_1)^2 + (L_2)^2 (Y_2)^2 ) + \mu(L_1)^2 (L_2)^2 (Y_1)^2 (Y_2)^2 \}|\). Here \(|(L_1)^2, (L_2)^2, (Y_1)^2, (Y_2)^2, |\) are the fast angles associated to the action \(|(L_1)^2, (L_2)^2, |\) i.e. the mean anomaly of \(|(E_1, E_2))|\) proportional to the area swept by \(|E_1, E_2))|\) from the focus to \(|(E_1, E_2))|\) and hence to the Keplerian time on \(|(E_1, E_2))|\).

Pairs of coplanar circles are singularities of this secular system and the study by Laplace of the corresponding linearized system gave the first result of (linear) stability of a planetary system (see [Las]), well before the establishment of the spectral theory of matrices: in this approximation, added to the fast motion of the planets on their respective ellipses (described by \(|(E_1, E_2))|\) there is a slow (secular) precession of the perihelia and the nodes of these ellipses (the *slow angles*) associated to small oscillations of their eccentricities and inclinations. As the averaged Hamiltonian does not depend any more on \(|(E_1, E_2))|\) the semi major axes of the ellipses do not vary in this approximation (this is Laplace's first *stability theorem*).

From Lindstedt series to K.A.M.

To go beyond, it is necessary to analyze the fate of the quasi-periodic motions just described under the remaining perturbation. Such perturbed quasi-periodic motions are given formally by the theory of *Lindstedt series*, whose existence was proved by Poincaré in the second volume of his epoch-making treatise *Les méthodes nouvelles de la mécanique céleste*. These series exist only when the unperturbed Keplerian frequencies are not in *resonance* (i.e. when their ratio is not a rational number) and they are generally divergent (Poincaré, *Méthodes Nouvelles*, chapter XIII). The breakthrough was made by Arnold (1961) who, developing a degenerate version of Kolmogorov's celebrated theorem (1954, the first letter of the K.A.M. acronym, which stands for Kolmogorov-Arnold-Moser), proved the existence of a set of positive measure of almost planar and almost circular quasi-periodic solutions when the ratio of the masses of the planets to the mass of the Sun is microscopic. Arnold's proof was complete only for the planar problem. In the spatial case, a new resonance is present: the trace of the linearized secular system is always zero. This fact, which generalizes the opposite motions of the perigee and the node in the secular system of the lunar problem is true in general for the spatial \(m_i\)-body problem. This was first noticed by Herman who gave a new proof of Arnold's theorem, valid in the spatial case for any number of planets. After the death of Herman, this proof was completely written down by Féjoz [F1]. Herman's resonance disappears when one reduces the rotational symmetry, and in fact P. Robutel had been able to complete Arnold's proof in the spatial three-body case thanks to the use of a computer for checking the non-degeneracy conditions.

Finally, the possibility of writing down long normal forms with the help of computers allows finding more realistic bounds for the masses to which KAM theory applies. Examples of this can be found in the works of L. Chierchia and A. Celletti on the Sun-Jupiter-Victoria system and those of A. Giorgilli and U. Locatelli on the Sun-Jupiter-Saturn system (*KAM Theory in Celestial Mechanics*).

The astronomer's three-body problem: ii) a caricature of the lunar problem

The motion of the Moon around the Earth can be considered in first approximation as a Keplerian motion perturbed by the action of the distant Sun. The perturbation here is more important than in the planetary case and the problem was the object of major works since Newton himself and, over half a century later, Clairaut, d'Alembert and Euler. More and more refined theories were given in particular by Laplace, Pontécoulant, Hansen, Delaunay, Hill, Adams, Brown..., describing more and more *inequalities* of the motion. A very nice history of the problem is given in [G] by M. Gutzwiller.

Incidentally, a global study of the secular system reveals how the planetary and lunar problems connect to each other and often have similar properties (see [F2]). Motivated by the work of Hill, where in first approximation the mass of the Moon is supposed to be zero, Poincaré, followed by Birkhoff, developed the so-called *restricted problem*, where the mass of the central body is zero, is Keplerian.

**The planar circular restricted problem:**
In the twentieth century, extensive search for families of periodic solutions in the restricted 3-body problem was accomplished, first by mechanical numerical exploration. The Jacobi constant, constructed by Bolotin, implies the existence of solutions with an erratic diffusion of the angular momentum and a much slower one of the eccentricities. A full orbit of the solution occurs when the Jacobi constant increases, the components of the Hill regions around the two non-zero masses merge, a case closer to the one of the true Moon. When the Jacobi constant is negative and large enough, everything is reduced to the study of the Poincaré's fixed point theorem, Moser's invariant curve theorem and Aubry-Mather theory prove respectively the existence of periodic motions of long period of the Moon around the Earth in the rotating frame, of quasi-periodic motions whose perigees have as envelope a smooth closed curve and of motions whose perigees have as envelope a Cantor set (closed curve with infinitely many holes). It is also possible to prove the existence of "stuttering orbits" as in figure 4, where the sign of the angular momentum changes from time to time and the solution comes arbitrarily close to collisions.

Higher values of the Jacobi constant
When the Jacobi constant increases, the components of the Hill regions around the two non-zero masses merge, a case closer to the one of the true Moon; this allows transit orbits which link neighborhoods of the two massive bodies (see [Sz]). In the animation of figure 6, the masses of S and E are respectively 0.9 and 0.1.

Periodic solutions
At the end of paragraph 36 of the first volume of the Méthodes nouvelles one reads Poincaré's famous sentence about periodic (or relatively periodic) solutions: "D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusque-là réputée inabordable. It is still unknown to day if periodic solutions of the three-body problem are dense in the bounded motions but their importance is unquestioned.

Poincaré's classification
In the planetary problem, Poincaré used various techniques, in particular continuation, to prove the existence of various families of periodic (or periodic modulo rotations) orbits. He defined sorts, genres, species. In the first sort, the eccentricities of the planets are small and they have no inclination; in the limit where the masses vanish, the orbits become circular, with rationably dependent frequencies. In the second sort, the inclinations are still zero but the eccentricities are finite; in the limit one gets elliptic motions with the same direction of major semi-axes and conjunctions or oppositions at each half-period. In the third sort, eccentricities are small but inclinations are finite and the limit motions are circular but inclined. Solutions of the second genre are called today subharmonics: they are associated to a given \( \mathcal{H} \)-periodic solution of one of the 3 sorts and their period is an integer multiple of \( \mathcal{H} \). Solutions of the second species are particularly interesting: in the limit of zero masses, the planets follow Keplerian orbits till they have a close encounter (i.e., in the limit, a collision) and then shift to another pair of Keplerian ellipses. A full symbolic dynamics of such almost collision orbits has been constructed by Bolotin: it implies the existence of solutions with an erratic diffusion of the angular momentum and a much slower one of the Jacobi constant.

Numerical exploration
In the twentieth century, extensive search for families of periodic solutions in the restricted 3-body problem was accomplished, first by mechanical exploration.
From a study of the conformal geometry of the shape sphere, R. Montgomery was able to deduce that any bounded zero angular momentum (and identification of the two hemispheres). Paths in the shape sphere have natural lifts to paths of three body configurations with zero angular momentum. See also [Mon2]. The (unoriented) similitude classes of triangles in $\mathbb{R}^3$ are in correspondence with a disk changes along a path in the shape sphere is given in $2\bar{z}_1z_2$.

The shape sphere is quotient of the unit sphere $|z_1|^2+|z_2|^2=1$ by the Hopf map $\lambda$ (notice that a stereographic projection has changed the sphere (minus one point) into a plane). The (unoriented) similitude classes of triangles in $\mathbb{R}^3$ are in correspondence with a disk (identification of the two hemispheres). Paths in the shape sphere have natural lifts to paths of three body configurations with zero angular momentum. From a study of the conformal geometry of the shape sphere, R. Montgomery was able to deduce that any bounded zero angular momentum (and

Stability, exponents, invariant manifolds

Poincaré initiated also the study of the stability of periodic orbits, introducing their characteristic exponents and their stable and unstable manifolds. The famous mistake in his 1889 prize memoir, where he had thought he had proved stability in the restricted problem, is about the intersection of these manifolds (see [BG]). Let us recall that the collinear relative equilibria are always linearly unstable; for the equilateral ones, linear stability occurs only when one mass greatly dominates the two others (Routh[5] criterion, see Trieste notes) as in the case of Sun-Jupiter-Trojans already alluded to. The study of the intersections of stable and unstable manifolds of known periodic solutions leads to the construction of orbits by methods of symbolic dynamics.

Minimizing the action

Solutions of the equations of motion $\ddot{x}=-\nabla U(x(t))$ are critical paths of the Lagrangian action $\int_0^T \left| \nabla U(x(t)) \right| dt$ defined on the Sobolev space $\Lambda=H^1([0,T],\mathbb{R}^3)$ of paths with value in the configuration space of the problem. Among these, the minimizers are likely to be the simplest and Poincaré proposed to look for them in a short note of 1896. Coercivity (i.e., forbidding minimizers at infinity) is achieved by restricting $\Lambda$ appropriately. Thanks to Tonelli's theorem this insures the existence of a minimizer with values in the closure of $\{x:|x|=1\}$ of $\{x:|x|=1\}$ that is of a path possibly with collisions. Indeed, Newton's force is weak enough so that the action remains finite along a path which ends in a configuration where some of the bodies are in collision. Technically, this comes from Sundman estimates $\{\|\dot{v}_i\|,\|v_j\|\}=O(|t-t_0|^{-\frac{1}{3}})$ for any pair of bodies entering a collision at time $t(t_0,\cdot)$ The equilateral homographic solutions (of any eccentricity) are characterized as action minimizers in their homotopy class among loops of fixed period $T$ in $\{x:|x|=1\}$ (Venturelli). The corresponding relative equilibrium is an action minimizer among period $T$ loops with the Italian symmetry $\{x:\left|\left\{\left|\dot{x}\right|=O(|t-T_0|^2)\right\}\right|=O(1)\}$. When the masses are all equal, the symmetries may interchange them. If the symmetry group contains a copy of $\{x:|x|=1\}$ acting by shifting the time by $T(3)$ and circularly permuting the bodies, one gets 3-body choreographies, for example the figure eight solution whose symmetry group is the dihedral group $D(6)$ of order 12. This last solution, first found numerically by C. Moore, was rediscovered and proved to exist by A. Chenciner and R. Montgomery (see [CM]); C. Simó showed its stability (see [CelMech]) and C. Marchal discovered its connection to the equilateral relative equilibrium through a Liapunov family of relative periodic solutions (see [Ma]). Animations of the figure eight solution and other choreographies appear in Bill Casselman's column. The symmetry groups of the planar three-body problem and the corresponding action-minimizing trajectories were classified by V. Barutello, D. Ferrario and S. Terracini in [BFT].

Global evolution

Lagrange-Jacobi and Sundman

The Lagrange-Jacobi identity $\ddot{t}x=-\nabla U(x(t))$ and the Sundman inequality $\int KJ^{2/3}ge(C)\,d^2\Sigma$ is the first tool for the analysis of the global evolution $\{x(t):t=0,|x(t)|=1\}$ where we suppose the galilean frame chosen so that the center of mass is fixed at the origin. The first (which implies the virial theorem) is an elementary derivation using the homogeneity of the potential, the second is a complex Cauchy-Schwarz inequality; when written as $\int (Kg,\dot{r})(J^2,\dot{r})\,d^2\Sigma$ it amounts to a comparison with a two-body problem obtained by computing the part of the velocity corresponding to homothetic deformation, miniaturizing the rotational part and forgetting the part which corresponds to deformation of the shape: the existence of a shape for a triangle is indeed a major difference with the two-body problem.

The shape sphere

Triangles in the plane modulo translations may be identified with points in $\mathbb{R}^3/\equiviv \mathbb{R}^3/\equiviv$ for example by choosing Jacobi coordinates $\lambda_{1,2}=\nabla \mathbb{R}^3/\equiviv \mathbb{R}^3/\equiviv$ where the conservation of energy implies infinite velocities at collision.
Collisions

Two important general results are due to Sundman (with precisions by Birkhoff for the second one): a triple (=total) collision can occur only if \( |C| = 0 \); more precisely, if \( |C| 
eq 0 \) and if the size \( \ell \) of the system becomes small enough, one body must escape to infinity. As in the two body problem, the escape is either parabolic \( \|\vec{r}_i(t)\| = O(t^{\frac{2}{3}}) \) or hyperbolic \( \|\vec{r}_i(t)\| = O(t) \). Painlevé proved that a singularity, i.e. a time after which a solution cannot be extended is necessarily a collision. In fact, double collisions may be regularized and this allowed Sundman to prove that if \( |C| \neq 0 \), solutions can be defined by series which converge for all values of a renormalized time, a result which unfortunately does not give any insight into the nature of these solutions. Near the end of his life, G. Lemaitre worked out a nice simultaneous regularization of the double collisions, associated with a 4-fold covering of the shape sphere ramified at the three double-collision points (see [Le]). Triple collisions are responsible for very complicated behaviour. Studying first the problem on the line \( (p=1) \) and regularizing the double collisions by bouncing, one sees that if a third body approaches near the center of mass immediately after a double collision, it can be ejected with arbitrarily high velocity (see [Mc]). The difference with a two-body collision is that now the entry and exit velocities of one body in a small ball around the center of mass can differ by an arbitrarily great amount. Energy conservation is no obstruction to such behavior since the two remaining bodies are left with a large, negative potential energy.

Final motions

The possible final motions of the system were analyzed by Chazy and later by Alexeiev (see [AKN], [Ma]). Particularly remarkable are the oscillatory solutions whose simplest model was given by Sitnikov for the spatial restricted problem: the two massive bodies describe almost circular Keplerian orbits in a plane while the zero mass body oscillates on the orthogonal line to the plane through the center of mass, the \( \lim \inf \) of its distance to it being finite, while the \( \lim \sup \) may be infinite (see [M] for a description of the symbolic dynamics associated to these solutions). For the planar problem \( (p=2) \), a whole set of complicated solutions was constructed by Moeckel using methods of symbolic dynamics (see [Mo]): the idea is to use the complicated dynamics near a triple collision, that is with small values of angular momentum; the resulting solutions pass near relative equilibria or escape solutions in any prescribed order. Technically it amounts to the existence of transversal heteroclinic solutions between singularities or periodic solutions of a regularized flow. Closer to mission design, heteroclinic solutions in the restricted problem have been used to save fuel by C. Simó and coauthors (see [Si], W.S. Koon, M.W. Lo, J.E. Marsden and S.D. Ross).... Astrophysicists extensively studied the complicated evolution of a binary star under successive parabolic or hyperbolic close encounters with a third star, each time different. This gravitational scattering plays an important role in the understanding of the evolution of stellar systems. Also much studied is the evolution of an initially bounded triple system with negative energy. The methods combine mathematical and physical considerations with extensive numerical simulations (see [HH] and [VK]).

The oldest open question in dynamical systems

According to Herman (see [H]), it is to determine if, in the conditions of the planetary case, with in particular \( |C| \neq 0 \) (which forbids triple collisions) and after regularization of double collisions, the non-wandering set of the flow is nowhere dense in an energy hypersurface. An affirmative answer would imply topological instability, the bounded orbits being nowhere dense, even if their measure can be positive when Arnold's theorem applies.

Non-integrability

The glimpse we just had of the complexity displayed by some classes of solutions of the three-body problem seems to indicate -- but does not prove in general -- the non-integrability of the problem. Indeed, several proofs of non-integrability have been given since the end of nineteenth century; they are in general not easy and rely on somewhat restrictive notions of non-integrability.

Bruns, Painlevé

The non-existence of first integrals algebraic in the Cartesian coordinates of the positions and the momenta other than the ones deduced from those
Poincaré

The result proved by Poincaré in the second volume of *Les méthodes nouvelles de la mécanique céleste* is of a different nature: it asserts the non-existence of new integrals which are uniform analytic functions in the elliptic elements and depend analytically on the (small) masses (or even admit a formal expansion in the masses with analytic coefficients). More precisely, Poincaré starts with a Hamiltonian of the form \( H_0 + i \mu H_{1}\), obtained in the study of the planetary problem; the series expansion is made with respect to the small parameter \( \mu \) which is of the order of the ratio of planetary masses to the mass of the Sun. If \( H_{1}(\mu) \) is a first integral, the vanishing of the Poisson bracket \( \{H,F\} \) implies constraints on the Fourier coefficients \( c_m(I) \) of \( H_1(I,\theta)=\sum c_m(I)e^{im\cdot\theta} \): these coefficients must vanish each time \( m \) is a resonance of \( H_0 \) (hence the importance of periodic solutions). In fact things are more complicated because of the degeneracy of \( H_0 \), which depends only on the fast actions. The theorem is a consequence of the fact, far from obvious, that enough Fourier coefficients of \( H_1 \) do not vanish. In contrast to Bruns’ result, Poincaré’s theorem does not say anything for any given choice of the masses.

Ziglin, Morales-Ramis

More recently, Ziglin’s and Morales-Ramis theories were used to prove the non-existence of additional meromorphic integrals in the neighborhood of well-chosen particular solutions: the basic idea here is to trace the implications of integrability on the structure of the differential Galois group of the variational equations along some explicitly known solution of the equations of motion (see in particular the works of A. Tsyvgintsev).

Two cases of integrability

- the secular system of the planetary (or lunar) planar three-body problem is four dimensional, hence completely integrable because the angular momentum is a first integral;
- if one replaces the Newtonian potential, inversely proportional to the square of the distance by the *Jacobi potential* inversely proportional to the square of the distance, a new first integral \( 2IH-J^2 \) of the N-body problem exists which was discovered by Jacobi. This implies the complete integrability of the three-body problem on the line with such a potential.

Still simpler than the 4-(and more)-body problem!

At least two important features appear when the number of bodies is greater than three:

- the possibility of superhyperbolic escape velocities and, associated to it, the existence of non-collision singularities, that is collision-free solutions where some bodies escape to infinity in finite time (Gerver for \( m \ge 3 \) bodies in the plane and \( m \ge 5 \) for five bodies in space, unknown for 4 bodies apart from the seminal work of Mather and McGehee on the line with double collisions regularized);
- While in the three-body problem, if \( \sqrt{|h||c|} \) is big enough, the integral manifold \( H=h<0, C=c \) is not connected (as in the restricted problem, the projection of the components on the configuration space are called Hill regions, see [Mo]) it becomes connected when the number of bodies increases.

Acknowledgments

The author is grateful to Rick Moeckel for providing the animations and to the chief editor for turning them slim enough so that they can enter the text. He thanks the following colleagues and the referees for helping him in various ways, from reading first drafts and correcting mistakes to giving advice: Alain Albouy, Martin Celli, Jacques Féjoz, Yanning Fu, Rick Moeckel, Laurent Niederman, Philippe Robutel, Marc Serrero, Nataliya Shcherbakova, Dima Treschev (Dima Treschev), Alexey Tsyvgintsev. Thanks to Douglas Heggie and Piet Hut for the permission to reproduce their figure and to Walter Craig for getting the permission to reproduce figures 4 and 5.

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Averaging

See also

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Sponsored by: Eugene M. Izhikevich, Editor-in-Chief of Scholarpedia, the peer-reviewed open-access encyclopedia
Reviewed by: Richard Moeckel, School of Mathematics, University of Minnesota, Minneapolis, MN
Reviewed by: Anonymous
Accepted on: 2007-10-03 09:26:33 GMT

Categories: Celestial mechanics | Dynamical Systems
Three-body problem: Three-body problem, in astronomy, the problem of determining the motion of three celestial bodies moving under no influence other than that of their mutual gravitation. No general solution of this problem (or the more general problem involving more than three bodies) is possible. As practically.