Mathematical platonists, awake! Here, in a book written to put the final nail in the coffin of platonism, we have the beginnings of a response to the one serious philosophical challenge to platonism: how can human beings, finite physical beings, ever develop an understanding of mathematics, abstract and infinite, disjoint from physical experience?

George Lakoff, a linguist, and Rafael Núñez, a cognitive psychologist, have long-standing interests in mathematics. They have written a unique and fascinating, if flawed, book attempting to "apply the science of mind to human mathematical ideas" (p. x) to discover where our mathematical ideas come from. This book introduces the discipline of "mathematical idea analysis:" how our understanding of human cognitive processes can account for the development of mathematical ideas.

Their thesis is that virtually all mathematical ideas arise as metaphors. Virtually all, because a bit of mathematics, called subtitizing — the ability to recognize very small numbers — involves innate capacities of our brains. But subtitizing cannot account even for arithmetic: while we can subitize numbers as large as 4, we can't subitize 4 + 3. For this we need metaphor. "Metaphor is not a mere embellishment; it is the basic means by which abstract thought is made possible. One of the principal results in cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts." (p. 39) Their metaphors, essentially isomorphisms, consist of a source domain, generally concrete; a target domain (the domain of the new objects being developed); and a mapping between the two. A very early example is Arithmetic As Object Collection (p. 54-55): the source domain is object collection, the target domain is arithmetic, collections of objects of the same size correspond to numbers, the size of the collection corresponds to the size of a number, the smallest possible collection to the number 1, etc.

Arithmetic As Object Collection is the simplest type of mathematical metaphor, a "grounding metaphor," using everyday experiences to ground abstract concepts such as addition. There are also "linking metaphors" which have both source and target domains within mathematics. An example is the metaphor Numbers Are Points on a Line (p. 279), with source domain points on a line, and target domain numbers; a point P corresponds to a number P', the origin to the number 0; a designated unit distance point I to the number 1; etc. More sophisticated mathematical ideas may involve several metaphors at once, in a "conceptual blend." For example, Dedekind's Number-Line blend "uses two metaphors: Spaces Are Sets and Numbers are Points on a Line." (p. 295)

Their most important metaphor is the Basic Metaphor of Infinity (BMI). They start with finite but continuous iterative processes, involving a beginning state, intermediate states, and a final resultant state, and map them to iterative processes that go on and on, but the final resultant state now becomes actual infinity. (p. 159) This is no longer quite an isomorphism. Which is, in some ways, a good thing: their earlier metaphors which are simply isomorphisms seem rather sterile. But now that we're getting something genuinely new, the ambiguity of how to go from the intermediate states to the final state leaves a gap that needs more explanation than they give. In particular, it can lead to paradoxes such as those exposed by Zeno, or one they discuss that involves the limit of arc-lengths of semicircles with centers at ((2m-1)/2,0), m=1,2,...,2^n-1: for each n, the semicircles' lengths add up to pi/2, yet the pointwise limit of the semicircles is the interval [0,1](p. 325 - 333).

Beginning with the BMI, they also start making rather frequent mathematical errors, particularly in their discussion of infinitesimals. For example, they develop the set of infinitesimals by constructing sets S(r) which are "the set of all numbers greater than zero and less than 1/r, satisfying the first nine axioms for the real numbers" (p. 228). Among these axioms (p. 200) are "the existence of identity elements for both addition and multiplication" (axiom 4) and the existence of additive (axiom 5) and multiplicative (axiom 6) inverses. Unfortunately, there is no set of real numbers greater than 0 and less than 1 containing identities and inverses: their sets are inconsistent! This can be fixed, of course, by being a bit more detailed. But it gets worse when they attempt to develop the "granular numbers", which are basically the smallest subfield of the hyperreal numbers generated by the real numbers and one infinitesimal. They construct this field by attempting to pick out "the first infinitesimal number produced by the BMI" (sic, p. 235) — but there can be no first such number! (They believe the granular numbers are a new
mathematical object that they've discovered, and go on, p. 254, to discuss why mathematicians could have been so blind as not to have found them sooner!!)

The book deteriorates from the BMI onward. In the chapter on "Real Numbers and Limits", the authors observe, correctly, that our usual epsilon-delta definition of limit doesn't really capture how we conceptualize limits, the process of a function approaching a limit, partly because we allow all possible real numbers as values for epsilon, resulting in acceptable epsilons which are irrelevant to the limit. They decide, therefore, to use a sequential definition of limit. However, rather than following a standard treatment of this topic (involving all subsequences of the sequence under consideration), they introduce the concept of a sequence of "critical elements," which are "those terms of the sequence that must converge in order for the sequence as a whole to converge" (p. 195); their notion is incoherent. A bit later they confuse how the standard definition of limit works, believing they can choose epsilons as they please (p. 199); similar errors continue through much of the remaining mathematical content.

Not satisfied with introducing a new field of intellectual inquiry, the authors devote the penultimate part of the book (the last part is an extended explanation, in terms of their metaphors, of the mathematical basis of $e^{\sqrt{3} + 1} = 0$) to the introduction of a new philosophy of mathematics which they believe to be implied by their conclusions. They assert that they have dealt a fatal blow to what they call the "Romanticism of Mathematics" (p. 339), roughly what is often referred to as platonism: "Mathematics is an objective feature of the universe ... What human beings believe about mathematics therefore has no effect on what mathematics really is. ... Since logic itself can be formalized as mathematical logic, mathematics characterizes the very nature of rationality. ..." As with many social constructivists (e.g., Reuben Hersh), they dislike this romance because "It intimidates people. ... It helps to maintain an elite and then justify it." (p. 341) Their arguments in favor of "human mathematics" are briefer and no more eloquent than those in Hersh's What is Mathematics. Really? and have little direct connection with the rest of the book. While I have no more sympathy than the authors for this elitism of mathematicians (and have devoted most of my life to undoing its effects), the elitism of mathematicians is no more a consequence of a belief that mathematical facts are an objective feature of the universe than the elitism of physicists is a consequence of their belief that physical facts are an objective feature of the universe. While their description of how humans develop concepts of mathematics is consistent with the restricted social constructivism of Hersh, it is also consistent with any reasonable version of platonism that distinguishes between mathematical facts and human knowledge of those mathematical facts.

One small annoyance: the references are broken into 6 categories: to find "Narayanan [1997]," you may have to look through them all before finding the full reference.

Despite its flaws, this book is a significant contribution to our understanding of mathematics' relation to people. Although the analysis has some defects, most first attempts to introduce a new discipline involve some important insights but also some stumbling around in the dark. The insights these authors introduce make at least the first half of the book well worth reading for anyone (advanced undergraduate and up) interested in the philosophy of mathematics, or in the genesis of mathematical ideas.

The authors' reply to this review appears as one of the "reader reviews" below.
that we must explain in cognitively plausible terms. Dynamic sequences is not a "decision" we make; rather it is the reality of how people think, a cognitive reality created idea: where? In what (cognitive) space? And so on. These are questions about a human understanding of a humanly questions like: From a cognitive perspective, what is approaching what? What is moving? From where to understand ideas such as "the limit of f(x) as x unending process that goes on and on. It is the BMI that brings forth this new metaphorical entity: an ending to an end. The BMI is the most important metaphor in the book, and she points out (also correctly) that the BMI is not an isomorphism (that's right!). Gold accurately observes that the Basic Metaphor of Infinity characterizes "something genuinely new" (i.e., an end to an unending process: actual infinity). Unfortunately, she seems not to understand what conceptual metaphors are and how they differ in kind from disembodied mathematical isomorphisms (which are literal, not metaphorical). As a result, she mistakenly claims that the BMI introduces an "ambiguity of how to go from the intermediate states to the final state", leaving "a gap that needs more explanation." It is incorrectly taking conceptual metaphors to be mathematical isomorphisms that generate that gap. Conceptual metaphors, being human cognitive mechanisms have many properties not captured by isomorphisms. As we say it explicitly in pages 45 and 46, "conceptual metaphors do not just map preexisting elements of the source domain onto preexisting elements of the target domain. They can also introduce new elements into the target domain" (italics in the book). These elements are not inherent to the target domain. In the BMI case, an end is not inherent: Unending process that goes on and on. It is the BMI that brings forth this new metaphorical entity: an ending to an unending process. Asking the question of how (exactly) to go from the intermediate states to the final state is a question that belongs to the realm of literal, not metaphorical processes. We give a simple example on page 46.

The moral here is this: It is totally consistent with what we know about human cognitive mechanisms that actual infinity could be a metaphorical idea. Via a specific conceptual metaphor (the BMI), an unending iterative process that goes on and on can be conceptualized as a process with an actual end and an actual final resultant state (which are precisely the elements not inherent to the target domain of unending processes).

Now, regarding the "mathematical errors" mentioned by Gold, she is in some extent right. There are some errors in the text of the first printing. But there are several things to say about them. First, several of these mistakes are editorial errors (not ours) that unfortunately affect mathematical content, which may mislead a careful reader, especially those who are mathematically trained (e.g., the definition of limits in p. 199). These editorial mistakes (and others) have been corrected in the errata section of the website for our book, which we set up soon after the publication of the book (www.unifr.ch/perso/nunezr). Second, there are indeed some passages in which we stated things in a sloppy way and we apologize (e.g., the characterization of the infinitesimals through the BMI state (which are precisely the elements not inherent to the target domain of unending processes). First, isomorphisms are explanatory constructs must satisfy strong biological and cognitive constraints. With a conceptual metaphor you have to explain data observed empirically, while with isomorphisms you don't necessarily have to. It is simply a confusion between disciplines to refer to conceptual metaphors as "isomorphisms." Our book is within the discipline of cognitive science and its subject matter is the cognitive science of mathematical ideas. To refer to conceptual metaphors as isomorphisms is to assume that mind and brain are disembodied mathematical isomorphisms, which is not. Our book is an attempt to give an account of mathematical ideas and inferences in terms of biologically and cognitively plausible mechanisms of the human mind, such as conceptual metaphors.

For example, the conceptual metaphor Arithmetic As Object Collection to which Gold refers is not a mere descriptive isomorphism. It is the embodied cognitive mechanism that gives an account of why the empirically observed expressions that exist in human (even technical) communication such as "three is bigger than two" or "four is smaller than eight" have the precise meaning they have, despite the fact that numbers in themselves don't have size. We believe that, because Gold misses the deep role of embodiment in our theoretical account throughout the book, she sees our characterization of "human mathematics" as "brief" and "not eloquent" enough, and having "little direct connection with the rest of the book."

We would also like to clarify some technical issues regarding how conceptual metaphors work. Gold says, correctly, that the Basic Metaphor of Infinity (BMI) is the most important metaphor in the book, and she points out (also correctly) that the BMI is not an isomorphism (that's right!). Gold accurately observes that the Basic Metaphor of Infinity characterizes "something genuinely new" (i.e., an end to an unending process: actual infinity). Unfortunately, she seems not to understand what conceptual metaphors are and how they differ in kind from disembodied mathematical isomorphisms (which are literal, not metaphorical). As a result, she mistakenly claims that the BMI introduces an "ambiguity of how to go from the intermediate states to the final state", leaving "a gap that needs more explanation." It is incorrectly taking conceptual metaphors to be mathematical isomorphisms that generate that gap. Conceptual metaphors, being human cognitive mechanisms have many properties not captured by isomorphisms. As we say it explicitly in pages 45 and 46, "conceptual metaphors do not just map preexisting elements of the source domain onto preexisting elements of the target domain. They can also introduce new elements into the target domain" (italics in the book). These elements are not inherent to the target domain. In the BMI case, an end is not inherent: Unending process that goes on and on. It is the BMI that brings forth this new metaphorical entity: an ending to an unending process. Asking the question of how (exactly) to go from the intermediate states to the final state is a question that belongs to the realm of literal, not metaphorical processes. We give a simple example on page 46.

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Gold’s misunderstanding of our goals and intentions can be more clearly seen through her interpretation of what we have called “granular numbers.” Nowhere in our book do we say that “the granular numbers are a new mathematical object that we have discovered” (how could we possibly make such a statement!). To begin with, “discovering” such a thing would be inconsistent with the non-platonic nature of embodied mathematics we endorse. What we say instead is that we have “invented the granulars by applying the BMI” (p. 254). This is overtly not a process within formal mathematics. It is the use of a cognitive process for creating mathematical ideas. In this case, we use everyday cognitive mechanisms such as the BMI together with the inferential structure of the idea of “speck” (which we hypothesize as the everyday idea that was the inspiration for Leibniz’s idea of infinitesimal).

It is in the realm of this cognitive exercise that we can affirm that the BMI produces “the first infinitesimal” (p. 235). This is not a mathematical result in the classic sense (i.e., it is not a result obtained by proving a mathematical theorem). Therefore it is not a “mathematical error” to say such a thing. “The first infinitesimal” is a consequence of the inferential structure of the BMI when applied to the particular case being discussed. The fact that this infinitesimal is the “first infinitesimal,” is an entailment of the metaphor, which generates a unique final resultant state, with no prior resultant state of that kind. That is why it is “first.” For all these reasons, Gold is incorrect when she says on page 254 that we “discuss why mathematicians could have been so blind as not to have found them [the granulars] sooner.” In that passage we ask an entirely different question: Why didn’t the remarkable mathematicians who have worked on the hyperreals develop such a number system when they could have done it easily? On the same page we give our answer, which doesn’t have anything to do with considering these mathematicians “so blind as not to have found them [the granulars] sooner.” That view simply does not fit with the respect bordering on awe that we have for those extraordinary mathematicians.

We hope these clarifications help mathematicians to understand (and to enjoy!) our book better. We are aware that it is not a simple task to try to follow a cognitive analysis of mathematical ideas when one has been trained as a professional mathematician, which is probably the case of most readers of this MAA column. We are also aware, and here we agree with Gold, that “most first attempts to introduce a new discipline involve some important insights but also some stumbling in the dark.” So, it is true that we are just starting this enterprise, and therefore there are still many unclear components. We see the cognitive science of mathematics as a multidisciplinary field, and we certainly need all the help we can get from professional mathematicians. We have to keep in mind, however, that our goal is to characterize mathematics in terms of cognitive mechanisms, not in terms of mathematics itself, e.g., formal definitions, axioms, and so on. Indeed, part of our job is to characterize how such formal definitions and axioms are themselves understood in embodied cognitive terms.

We simply have a different job than professional mathematicians have. We have to answer such questions as: How can a number express a concept? How can mathematical formulas and equations express general ideas that occur outside of mathematics, ideas like recurrence, change, proportions, self-regulating processes, and so on? How do ideas within mathematics differ from similar (but not identical) ideas outside mathematics (e.g., the idea of “space” or “continuity”)? How can “abstract” mathematics be understood? What cognitive mechanisms are used in mathematical understanding?

We hope that such questions asked and answered from outside of mathematics proper will interest mathematicians. And we hope that they will not be mistaken for questions and answers within mathematics.

Finally, we want to thank Bonnie Gold once more for her review, and to thank the MAA online book review column for finding a reviewer of such mathematical competence and with the openness and energy required for such an undertaking. We have the greatest respect for her. We are all too aware of how much effort goes into such a review and how misunderstandings can arise naturally across disciplines. It is only through a forum such as this that such issues can be aired in a spirit of cooperation and honest inquiry.

George Lakoff and Rafael E. Núñez

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